

All-phononic Digital Transistor on the Basis of Gap-Soliton Dynamics in Anharmonic Oscillator Ladder

Merab Malishava, Ramaz Khomeriki

Physics Department, Javakishvili Tbilisi State University, 3 Chavchavadze, 0179 Tbilisi, Georgia

A conceptual mechanism of amplification of phonons by phonons on the basis of nonlinear band-gap transmission (supratransmission) phenomenon is presented. As an example a system of weakly coupled chains of anharmonic oscillators is considered. One (source) chain is driven harmonically by boundary with a frequency located in the upper band close to the band edge of the ladder system. Amplification happens when a second (gate) chain is driven by a small signal in the counter phase and with the same frequency as first chain. If the total driving of both chains overcomes the band-gap transmission threshold the large amplitude band-gap soliton emerges and amplification scenario is realized. The mechanism is interpreted as nonlinear superposition of evanescent and propagating nonlinear modes manifesting in a single or double soliton generation working in band-gap or band-pass regimes, respectively. The results could be straightforwardly generalized for all-optical or all-magnonic contexts and has all the promises for logic gate operations.

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Since the celebrated Fermi-Pasta-Ulam (FPU) first numerical experiment [1] in 1954, anharmonic oscillator chains became a powerful tool in dealing with both fundamental aspects of statistical physics [2, 3] and nonlinear wave phenomena [4] and, at the same time, serve as the simplest prototypes for extremely complex condensed matter systems [5, 6] and even biophysical processes [7, 8]. In particular, studies on the FPU chains together with its further developments, namely nonintegrable (Klein-Gordon [9] and Frenkel-Kontorova [10]) and integrable Toda [11] chains, had an impact on the discovery of solitons [12, 13], helped much in understanding of interplay between integrability and chaos [14], have been widely applied for understanding of anomalous thermal conduction and rectification properties in realistic physical systems [15–18], have applied to describe transport properties in electric transmission lines [23] and even in quantum systems, such as Josephson junction parallel arrays and lattices [24, 25], and until now are widely used to resolve thermal equipartition issues [26].

Surprisingly, the ladder extension of anharmonic one-dimensional systems is rarely studied (but see Refs. [27, 28]), although there exists a wide range of applications for realistic systems, e.g. optical directional couplers [29, 30], weakly coupled classical [19] or quantum [20] spin chains, coupled two or multicomponent systems [21, 22], etc. In the present letter we aim to consider two weakly coupled FPU chains in order to realize digital all-phononic amplification of acoustic signals. The considered concept of amplification could be straightforwardly extended in case of similar all-optical [31, 32] and all-magnonic [33, 34] devices.

Phonon laser [35, 36] developments renew the interest in various applications of monochromatic acoustic waves. Several ideas have been proposed for phonon diodes [37–43] and all-phononic transistors [44, 45] working on magneto-acoustic, nonlinear wave-mixing or mode-mode

interaction effects.

In this letter we implement a nonlinear band-gap transmission mechanism [46–48] producing gap-solitons in order to achieve digital amplification of weak acoustic signals. All three ports of the proposed device work on a single operational frequency and the schematics is presented in Fig. 1a, where the amplifying part of the device is indicated by a dashed frame. Green and blue color chains outside the frame are used for supplying the sig-

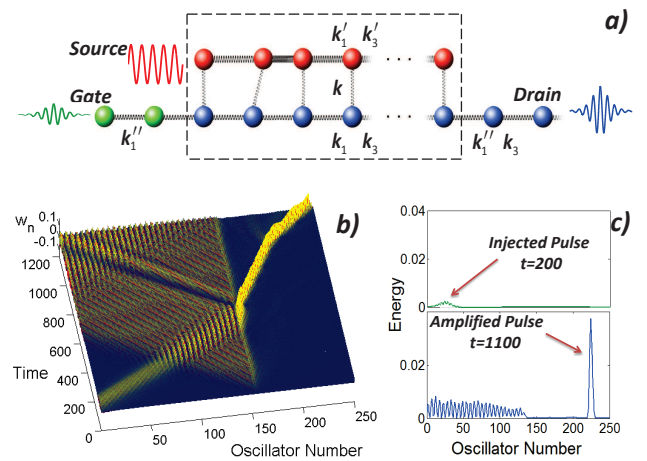


FIG. 1: a) The conceptual scheme of phonon transistor with indications of source and gate signal forms and supply places. Amplified signal is monitored at the Drain port. b) Space-time dynamics of gap-soliton creation and propagation in the lower chain, where in the range $1 < n < 150$ one has a linear chain (green balls in the upper graph) and for $150 < n < 250$ soliton propagation occurs in a nonlinear chain (blue balls). The upper chain is coupled with the lower one in the range $150 < n < 200$. c) Energies of the input Gate signal (green curve) and the amplified gap-soliton output Drain signal (blue curve) monitored in the same lower chain

nal at the gate and monitoring the output pulse at the Drain ports, respectively. The signal is injected from the left (gate) linear oscillator chain (green balls). The gate signal has a carrier frequency within the band gap of the system of the nonlinear oscillator ladder (red and blue balls inside the frame) and without source driving cannot propagate further. At the source input we apply large amplitude harmonic driving with the same frequency as a gate signal. The amplitude of the source is just below the band gap transmission threshold for antisymmetric mode of the ladder and together with the gate signal the overall amplitude is enough to exceed the threshold and a single large amplitude soliton passes the ladder system and appears at the drain. While without the gate signal the soliton is not produced. Thus it is clear that the soliton amplitude is mostly defined by source driving and the digital amplification scenario takes place. Main results of the numerical simulations are presented in Fig. 1b and c). Particularly, graph (b) describes space-time evolution of the displacements w_n of the lower chain, while in graph (c) the energies of signals at the Gate and Drain ports are displayed. For clarity we present a movie file in Supplemental Material (SM) in order to show the signal propagation and distribution among upper and lower chains.

For the analytical consideration we examine the ladder part of the system (blue and red balls within the frame in Fig. 1a) modeling the system as two weakly coupled FPU chains as follows:

$$\begin{aligned} m\ddot{u}_n &= k'_1(u_{n+1} + u_{n-1} - 2u_n) + k'_3(u_{n+1} - u_n)^3 \\ &\quad + k'_3(u_{n-1} - u_n)^3 + k(w_n - u_n) \\ m\ddot{w}_n &= k_1(w_{n+1} + w_{n-1} - 2w_n) + k_3(w_{n+1} - w_n)^3 \\ &\quad + k_3(w_{n-1} - w_n)^3 + k(u_n - w_n) \end{aligned} \quad (1)$$

where u_n and w_n stand for displacements of the n -th oscillators (with mass m) of the upper and lower chains respectively; k, k'_1, k_1, k_3 and k'_3 are linear and nonlinear coefficients of stiffness of the springs. Without restricting generality we rescale displacement amplitudes and time such that the parameters of the upper chain take the unit values $m = k'_1 = k'_3 = 1$. All numerical simulations will be done using this scaling and fixing the parameters of the lower chain and interchain coupling as follows: $k_1 = 1.1, k_3 = 3.5, k = 0.2$. We apply dirichlet boundary condition at the left end of both chains oscillating the balls $n = 0$ of upper and lower chains with the source and gate amplitudes, respectively.

In the linearized version of (1) we can readily define in the n -th site of the ladder a two component vector (u_n, w_n) and seek for a solution in a form of harmonic waves

$$(u_n, w_n) = (R, 1)e^{i(pn - \Omega t)} + c.c., \quad (2)$$

which gives us two branches of antisymmetric mode (neighboring the n -th sites in different chains of the lad-

der oscillate in counter phase) and symmetric mode (the n -th sites oscillate in the same phase). Modes with corresponding dispersion relations $\Omega_1(p)$ and $\Omega_2(p)$ with maximum values at $p = \pi$ are displayed in Fig. 2. Those modes are characterized by respective components R_1 and R_2 given by the following formulas

$$R_j = k/[k - (\Omega_j)^2 + 2(1 - \cos p_j)] \quad (3)$$

where $j = 1, 2$. In the case of identical chains in the ladder $R_j = \pm 1$, i.e. the oscillation amplitudes of interchain neighbor oscillators are the same, while in our case of asymmetric ladder the oscillation amplitudes are larger in the lower chain (for $j = 1$) or in the upper chain ($j = 2$). If one drives the ladder with a monochromatic frequency Ω the excitation wavenumbers p_j of the respective modes are calculated via the relations:

$$\Omega = \Omega_1(p); \quad \Omega = \Omega_2(p), \quad (4)$$

Depending on the driving frequency (see Fig. 2) the following three cases could be realized: 1) for excitation frequencies $\Omega < \Omega_2(\pi) < \Omega_1(\pi)$ a nonlinear wave enters the system, then separates into two soliton waves; 2) for excitation frequencies $\Omega_2(\pi) < \Omega < \Omega_1(\pi)$ a nonlinear wave generates a single soliton associated to the antisymmetric mode $j = 1$; And finally, 3) for the $\Omega > \Omega_1(\pi) > \Omega_2(\pi)$ a nonlinear wave cannot enter the system unless the driving amplitude exceeds the band gap transmission threshold.

We start our analysis from considering driving frequencies, which are within the band of both modes (the lowest dashed horizontal line in Fig. 2). Then the wavenumbers p_j of both modes are real and could be found solving Eqs. (4). Then a weakly nonlinear solitonic solution could be

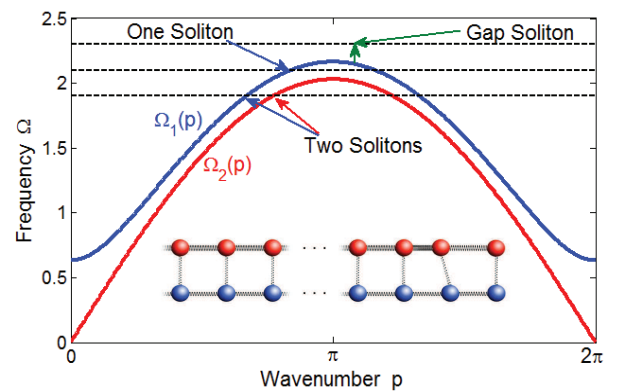


FIG. 2: Red and blue curves represent linear dispersion relations of the antisymmetric $\Omega_1(p)$ and symmetric $\Omega_2(p)$ branches, respectively. Dashed horizontal lines represent three different cases ($\Omega = 1.90, \Omega = 2.10$ and $\Omega = 2.18$) which lead to the different kind of nonlinear wave transport in the system. Inset shows the schematics of the system of the FPU ladder.

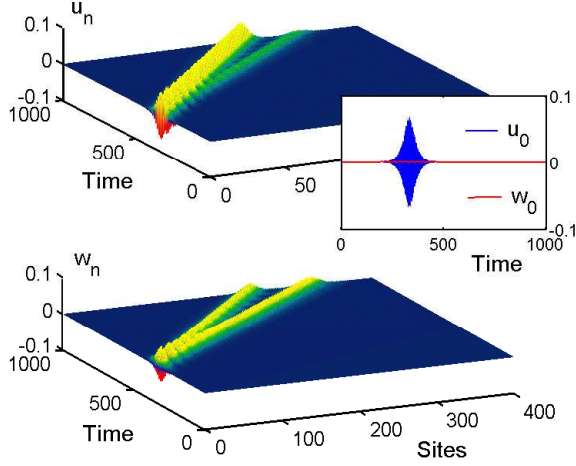


FIG. 3: Numerical simulations on the FPU ladder (1) with the boundary driving according to (10): the upper chain (upper graph) is driven with a frequency $\Omega = 1.9$ and the lower chain (lower graph) is kept pinned. Inset shows how the ultimate left $n = 0$ balls of the ladder are driven in time.

presented as a modulation of harmonic expression (2) of the corresponding mode [49–51]:

$$(u_n^j, w_n^j) = (R_j, -1) e^{i(p_j n - \Omega_j t)} \varphi_j(\xi, \tau) + c.c., \quad (5)$$

where $\varphi_j(\xi, \tau)$ is a function of slow variables $\xi = \epsilon(n - v_j t)$ and $\tau = \epsilon t^2$, where $v_j = \partial \Omega_j(p) / \partial p|_{p=p_j}$ is a group velocity of the respective mode and ϵ is a small expansion parameter. At the same time $\varphi_j(\xi, \tau)$ obeys the nonlinear Schrödinger (NLS) equation (please see for details SM file):

$$2i \frac{\partial \varphi_i}{\partial \tau} + \Omega_i'' \frac{\partial^2 \varphi_i}{\partial \xi^2} + \Delta_i |\varphi_i|^2 \varphi_i = 0 \quad (6)$$

with the following parameters:

$$\Delta_j = \frac{12(1 - \cos p_j)^2 (k_3 + R_j^4)}{\Omega_j (1 + R_j^2)}, \quad \Omega_j'' = \frac{\partial^2 \Omega_j}{\partial p^2} \Big|_{p=p_j}, \quad (7)$$

and finally one arrives to the soliton solution of the respective mode as follows:

$$(u_n^j, w_n^j) = (R_j, -1) \frac{A_j \cos(\Omega t - p_j n)}{\cosh[(n - v_j t)/\Lambda_j]} \quad (8)$$

where A_j is a soliton amplitude, while soliton width Λ_j and modified dispersion relation are given by:

$$\Lambda_j = \frac{1}{A_j} \sqrt{\frac{2\Omega_j''}{\Delta_j}}, \quad \Omega = \Omega_j + \frac{1}{4} \Delta_j A_j^2. \quad (9)$$

Let us note that in the nonlinear case the latter relation has to be applied for a computation of soliton carrier wavenumber p_j .

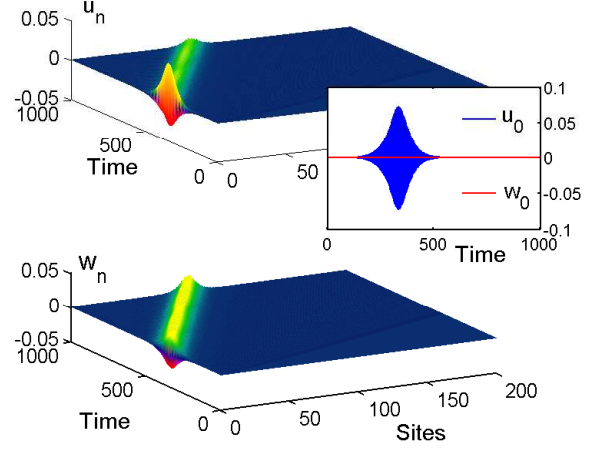


FIG. 4: Same as in Fig. 3, but now the upper chain is driven by a frequency $\Omega = 2.1$ and the lower chain is again pinned.

In weakly nonlinear limit (small soliton amplitudes $A_j \ll 1$) and large relative group velocities $|v_1 - v_2|/v_{1,2} \gtrsim 1$ one can combine the solutions (8) acquiring additional phase shift [50] which could be safely neglected in the mentioned limits. By this one is able to construct the solution, which describes the initial excitation of the boundary of the solely upper chain. In particular, if one takes $A_1 = A_2$ and finds such an excitation frequency that $v_1/\Lambda_1 = v_2/\Lambda_2$, the combination $(u_n^1, w_n^1) - (u_n^2, w_n^2)$ at the origin $n = 0$ gives

$$(u_0^1, w_0^1) - (u_0^2, w_0^2) = (R_1 - R_2, 0) \frac{A_1 \cos(\Omega t)}{\cosh[v_1 t/\Lambda_1]}, \quad (10)$$

thus driving both chains in time according to the above expression one can excite two soliton solution belonging to different branches. That is displayed in Fig. 3, driving in numerical simulations the left end of the upper chain u_0 with a frequency $\Omega = 1.90$ and amplitude $A = (R_1 - R_2) A_1$ with $A_1 = 0.025$ and calculating R_j from Eq. (3). At the same time the lower chain is kept pinned at the left boundary ($w_0 = 0$) according again to the expression (10). As seen, the numerical test is just in tact with the expectation, as far as according to (8) we observe different amplitudes for the solitons in the upper chain and just the same $A_1 = 0.025$ in the lower one.

Next we examine one soliton generation driving again only upper chain with a frequency lying in the limits $\Omega_2(\pi) < \Omega < \Omega_1(\pi)$, particularly we apply $\Omega = 2.10$ in numerical simulations (see middle horizontal line in Fig. 2). In this case antisymmetric mode ($j = 1$) solution could be again presented in solitonic form (8), while the symmetric mode ($j = 2$) has no longer a solitonic profile, instead it is described by evanescent wave since the corresponding wavenumber p_2 is imaginary number (solution of dispersion relation $\Omega = \Omega_2(p)$ has no real roots):

$$(u_n^2, w_n^2) = (R_2, -1) B(t) e^{-|p_2|n} \cos(\Omega_2 t) \quad (11)$$

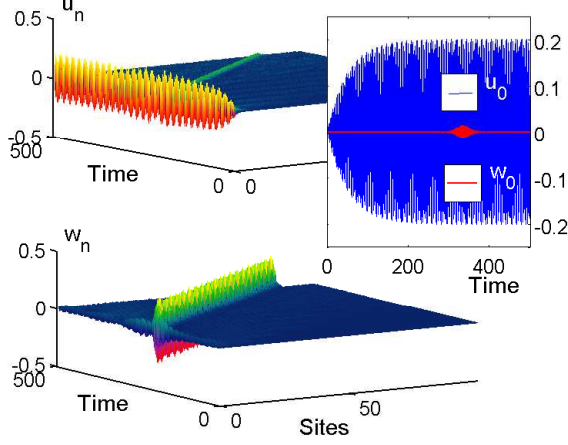


FIG. 5: The amplification scenario for FPU ladder. In the upper chain (upper panel) we provide the continuous driving of the left end with an amplitude just below the threshold (12) and the band-gap frequency $\Omega = 2.18$. While the left end of the lower chain (lower panel) is perturbed by a small amplitude signal with the same carrier frequency. Inset shows how the ultimate left $n = 0$ balls of the ladder are driven in time.

where $B(t)$ can slowly vary in time. This means that we observe only one soliton entering the chain. As we try to nullify oscillations in the lower chain $B(t)$ should take a form of $B(t) = A_1 \text{sech}(v_1 t / \Lambda_1)$ and then the combination $(u_n^1, w_n^1) - (u_n^2, w_n^2)$ at the origin $n = 0$ gives the same form of the driving as in the previous case (10) of the two soliton generation. The results are displayed in Fig. 4, and as seen driving the upper chain with a frequency $\Omega = 2.10$ now one monitors the generation of a single envelope soliton.

Finally we consider the case $\Omega = 2.18$ (upper dashed line in Fig. 2) lying in the band gap of both modes, for which only evanescent wave solutions (11) is realized for the modes if the driving amplitude is small. However, if the amplitude exceeds some threshold value, a gap soliton can be created and propagate along the ladder. For the estimation of this threshold value, we assume that the upper chain is driven with the amplitude A while the lower one is kept pinned. Then, looking at the typical solution of such a scenario (10) one can notice that the weight of the antisymmetric mode A_1 is defined from the relation $A = A_1 (1 - R_1/R_2)$ and the threshold value is calculated from the expression of nonlinear frequency shift (9):

$$A^{th} = (1 - R_1/R_2) \sqrt{4[\Omega - \Omega_1(\pi)]/\Delta_1}. \quad (12)$$

Determining A^{th} gives us an opportunity to realize the amplification scenario. For this we create the continuous driving in the upper chain with a band-gap frequency $\Omega = 2.18$ and amplitude just below the threshold, then even small counter-phase pulse in the lower chain can

help to overcome the threshold and provide the necessary amplification effect for the weak pulse. For the numerical experiment displayed in the Fig. 5 we use a continuous driving with the amplitude $A = 0.202$, while the pulse amplitude in the lower chain can be of the order of 0.015. As seen such a small pulse is enough to create a gap soliton and realize amplification scenario in the oscillator ladder. In order to provide a realistic input-output ports we have lengthened lower chain adding linear part at the left and nonlinear part at the right (see Fig. 1a) and as could be seen the results are in agreement with developed analytical scheme.

This amplification mechanism could be directly verified using cantilever arrays [19], particularly, one can examine two coupled in parallel cantilever arrays and use the scenario presented in the Fig. 1. Note that although a cantilever array model include onsite coupling terms in contrast to our model of the FPU lattices, the consideration of the amplification mechanism will be the same, since we consider upper band-gap localized modes (staggered excitations) which are similar in both types of the anharmonic chains. Using the soft mono-element lattice parameters from [19] and taking interchain coupling constant as 20% from onsite coupling coefficient it follows that the upper band-gap, appearing due to discreteness, starts at $\Omega_1(\pi) = 116.8\text{KHz}$ and applying the driving in the band-gap with a frequency $\Omega = 117.8\text{KHz}$ one can calculate the threshold amplitude for the band-gap transmission according to (12) and it gives the value $A_{th} = 0.4\mu\text{m}$ that is much less than the typical distance between cantilever and substrate $d = 10\mu\text{m}$ and excitation can propagate through the array. The width of the soliton appearing after the amplification could be calculated from (9) and gives the value $\Lambda_1 \approx 10$ lattice sites. Thus it is very clear that the developed amplification scheme is very robust with respect to the model choice [52].

Concluding, it should be emphasized that we are using a single operational frequency and thus the output signal could be readily used for the further processing. Suggested mechanism could be applied to study amplification in quantum systems, e.g. for trapped cold atoms in optical lattice ladders. In this case engineering edge defect site one can reach the threshold effect by tuning the atomic onsite interaction strength. Then above some threshold value quantum solitons will be created via resonance process between the edge defect and the quantum bound state modes. But this issue needs a further detailed investigation.

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SUPPLEMENTAL MATERIAL

We start from the analysis of two weakly coupled FPU chains displayed as Eq. (1) in the main text of the manuscript:

$$\begin{aligned} m\ddot{u}_n &= k'_1(u_{n+1} + u_{n-1} - 2u_n) + k'_3(u_{n+1} - u_n)^3 \\ &\quad + k'_3(u_{n-1} - u_n)^3 + k(w_n - u_n) \\ m\ddot{w}_n &= k_1(w_{n+1} + w_{n-1} - 2w_n) + k_3(w_{n+1} - w_n)^3 \\ &\quad + k_3(w_{n-1} - w_n)^3 + k(u_n - w_n) \end{aligned} \quad (13)$$

According to a well established procedure [49–51] of multi-scaling approach we are seeking a weakly nonlinear solution of (13) in a form of following perturbative expansion:

$$\mathbf{U} = \sum_{\alpha=1}^{\infty} \epsilon^{\alpha} \sum_{m=-\infty}^{+\infty} \mathbf{U}_m^{(\alpha)}(\tau, \xi) e^{im(pn - \Omega t)} \quad (14)$$

where we define column vector $\mathbf{U}^{(\alpha)} = (u_n^{(\alpha)}, w_n^{(\alpha)})$, while ξ and τ are slow variables introduced through: $\xi = \epsilon(n - vt)$ and $\tau = \epsilon t^2$; v is a soliton group velocity defined below and ϵ is a small expansion parameter.

We go on with equating powers of ϵ substituting expansion (14) in set of equations (1). In the linear approximation we have the column vector $\mathbf{U}_1^{(1)} \equiv (u_n^{(1)}, w_n^{(1)}) = \varphi(\xi, \tau)\mathbf{R}$ and $\mathbf{U}_m^{(1)} = 0$ for $|m| \neq 1$; not restricting generality we can take a space-time independent column vector as $\mathbf{R} = (R, 1)$, where R is a complex number and $\varphi(\xi, \tau)$ is a scalar function of slow variables to be determined in the next approximations. Then by considering $\alpha = 1$ (linear approximation) and the harmonic $m = 1$ we arrive to the equation:

$$\hat{\mathbf{W}} * \mathbf{R} = 0 \quad (15)$$

where

$$\hat{\mathbf{W}} = \begin{pmatrix} \Omega^2 + 2(\cos p - 1) - k & k \\ k & \Omega^2 + 2k_1(\cos p - 1) - k \end{pmatrix}$$

the solvability of which demands $\text{Det}(\hat{\mathbf{W}}) = 0$, which gives us two branches of dispersion relations:

$$\begin{aligned} \Omega_{1,2}^2 &= (1 - \cos p)(1 + k_1) + k \pm \\ &\quad \pm \sqrt{(1 - \cos p)^2(1 - k_1)^2 + k^2} \end{aligned} \quad (16)$$

and two corresponding column vectors $\mathbf{R}_j = (R_j, 1)$ with R_j expressed with the linear parameters of the problem $R_j = k/[k - \Omega_j^2 + 2(1 - \cos p)]$, where $j = 1, 2$. Next we introduce a row vector $\mathbf{L} = (L, 1)$ through the equation $\mathbf{L} * \hat{\mathbf{W}} = 0$, that gives us two row vectors \mathbf{L}_j . In our case the respective components of row \mathbf{L}_j and column \mathbf{R}_j are identical $L_j = R_j$. Thus in linear limit we have following matrix relations:

$$\hat{\mathbf{W}}(\Omega_j) * \mathbf{R}_j = 0, \quad \mathbf{L}_j * \hat{\mathbf{W}}(\Omega_j) = 0. \quad (17)$$

In the following for presentation clarity we omit the indexes j and restore them at the end of the calculations.

We go on with a second approximation ($\alpha = 2$) substituting again (14) into (13) and considering first harmonic $m = 1$, which leads us to the following equation:

$$\hat{\mathbf{W}}\mathbf{U}^{(2)} + 2i(\hat{\mathbf{B}} - \Omega v \hat{\mathbf{I}}) \frac{\partial \varphi}{\partial \xi} \mathbf{R} = 0, \quad (18)$$

where

$$\hat{\mathbf{B}} = \begin{pmatrix} \sin p & 0 \\ 0 & k_1 \sin p \end{pmatrix} \quad (19)$$

Then multiplying (18) by \mathbf{L} one has

$$\mathbf{L}(\hat{\mathbf{B}} - \Omega v \hat{\mathbf{I}}) \frac{\partial \varphi}{\partial \xi} \mathbf{R} = 0. \quad (20)$$

In order to identify constant v in the equation above, let us take the derivative of (15) over p and multiply then on the row vector \mathbf{L} . One gets:

$$\mathbf{L} \frac{\partial \hat{\mathbf{W}}}{\partial p} \mathbf{R} = 2\mathbf{L} \left(\frac{d\Omega}{dp} \hat{\mathbf{I}} - \hat{\mathbf{B}} \right) \mathbf{R} = 0. \quad (21)$$

Comparing now (20) and (21) we immediately get the equality $v = \partial \Omega / \partial p$, thus the definition for group velocity, while from (18) one can solve $\mathbf{U}^{(2)}$ as follows:

$$\mathbf{U}^{(2)} = -2i\hat{\mathbf{W}}^{-1}(\hat{\mathbf{B}} - \Omega v \hat{\mathbf{I}}) \frac{\partial \varphi}{\partial \xi} \mathbf{R}. \quad (22)$$

In the third approximation, equating powers of ϵ for $\alpha = 3$ and first harmonic $m = 1$ we have:

$$\begin{aligned} \hat{\mathbf{W}}\mathbf{U}^{(3)} + 2i(\hat{\mathbf{B}} - \Omega v \hat{\mathbf{I}}) \frac{\partial \mathbf{U}^{(2)}}{\partial \xi} + 2i\Omega \frac{\partial \varphi}{\partial \tau} \mathbf{R} - \\ - \hat{\mathbf{C}} \frac{\partial^2 \mathbf{U}^{(1)}}{\partial \xi^2} + 12(1 - \cos p)^2 \mathbf{N} |\varphi|^2 \varphi = 0 \end{aligned} \quad (23)$$

where

$$\hat{\mathbf{C}} = \begin{pmatrix} v^2 - \cos p & 0 \\ 0 & v^2 - k_1 \cos p \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} R^3 \\ k_3 \end{pmatrix} \quad (24)$$

Now noting that

$$2(\hat{\mathbf{B}} - \Omega v \hat{\mathbf{I}}) \equiv -\frac{\partial \hat{\mathbf{W}}}{\partial p}; \quad \hat{\mathbf{C}} \equiv \frac{1}{2} \frac{\partial^2 \hat{\mathbf{W}}}{\partial p^2} - \Omega \frac{\partial^2 \Omega}{\partial p^2} \hat{\mathbf{I}} \quad (25)$$

We can further simplify (24) multiplying it on \mathbf{L} and taking into account (22) and (25):

$$\begin{aligned} \mathbf{L} \left(\Omega \frac{\partial^2 \Omega}{\partial p^2} \hat{\mathbf{I}} + \frac{\partial \hat{\mathbf{W}}}{\partial p} \hat{\mathbf{W}}^{-1} \frac{\partial \hat{\mathbf{W}}}{\partial p} - \frac{1}{2} \frac{\partial^2 \hat{\mathbf{W}}}{\partial p^2} \right) \mathbf{R} \frac{\partial^2 \varphi}{\partial \xi^2} \\ + 2i\Omega \frac{\partial \varphi}{\partial \tau} \mathbf{L} \mathbf{R} + 12(1 - \cos p)^2 \mathbf{L} \mathbf{N} |\varphi|^2 \varphi = 0 \end{aligned} \quad (26)$$

We can get a final form for (26) taking first and second derivatives of Eq. (15) over p :

$$\begin{aligned} \frac{\partial \hat{\mathbf{W}}}{\partial p} \mathbf{R} + \hat{\mathbf{W}} \frac{\partial \mathbf{R}}{\partial p} &= 0; \\ \frac{\partial^2 \hat{\mathbf{W}}}{\partial p^2} \mathbf{R} + 2 \frac{\partial \hat{\mathbf{W}}}{\partial p} \frac{\partial \mathbf{R}}{\partial p} + \hat{\mathbf{W}} \frac{\partial^2 \mathbf{R}}{\partial p^2} &= 0 \end{aligned} \quad (27)$$

Solving now $\partial \mathbf{R} / \partial p$ from the first equation and substituting it in the second one and then multiplying it on \mathbf{L} one gets the following relation:

$$\mathbf{L} \frac{\partial \hat{\mathbf{W}}}{\partial p} \hat{\mathbf{W}}^{-1} \frac{\partial \hat{\mathbf{W}}}{\partial p} \mathbf{R} - \frac{1}{2} \mathbf{L} \frac{\partial^2 \hat{\mathbf{W}}}{\partial p^2} \mathbf{R} = 0, \quad (28)$$

and now substituting this into the (26) and restoring indexes j -s one finally arrives to the Nonlinear Schrödinger (NLS) Equation for two nonlinear modes $j = 1, 2$:

$$2i \frac{\partial \varphi_j}{\partial \tau} + \Omega_j'' \frac{\partial^2 \varphi_j}{\partial \xi^2} - \Delta_j |\varphi_j|^2 \varphi_j = 0 \quad (29)$$

where

$$\Delta_j = \frac{12(1 - \cos p_j)^2 (k_3 + R_j^4)}{\Omega_j (1 + R_j^2)}, \quad \Omega_j'' = \left. \frac{\partial^2 \Omega_j}{\partial p^2} \right|_{p=p_j} \quad (30)$$

and wavenumbers p_j are the solutions of respective dispersion relations $\Omega = \Omega_j + \Delta_j A_j^2 / 4$.

We use the same approach considering cantilever arrays. Beginning with modified equations of motion:

$$\begin{aligned} m \ddot{u}_n + k'_{20} u_n + k'_{40} u_n^3 - k'_1 (u_{n+1} + u_{n-1} - 2u_n) - \\ - k'_3 (u_{n+1} - u_n)^3 - k'_3 (u_{n-1} - u_n)^3 + k(u_n - w_n) &= 0 \\ m \ddot{w}_n + k_{20} u_n + k_{40} u_n^3 - k_1 (w_{n+1} + w_{n-1} - 2w_n) - \\ - k_3 (w_{n+1} - w_n)^3 - k_3 (w_{n-1} - w_n)^3 + k(w_n - u_n) &= 0 \end{aligned}$$

In order to estimate the effect we take the following approximate values of the problem parameters: $m = 10^{-12} \text{ kg}$, $k_1 = k'_1 = k_{20} = k'_{20} = 0.1 \text{ kg/s}^2$, $k_3 = k'_3 = 10^{10} \text{ kg/s}^2 \text{ m}^2$, $k_{40} = k'_{40} = 10^8 \text{ kg/s}^2 \text{ m}^2$, and take weak interchain linear coupling coefficient as $k = 0.02 \text{ kg/s}^2$. Then the frequencies of two branches Ω_1 and Ω_2 are the solutions of matrix dispersion relation

$$\text{Det} \begin{bmatrix} \Omega^2 + 2k'_1 (\cos p - 1) - k - k'_{20} & k \\ k & \Omega^2 + 2k_1 (\cos p - 1) - k - k_{20} \end{bmatrix} = 0$$

and from the ordinary procedure developed above we again get NLS equations with following nonlinear coefficients for the antisymmetric ($j = 1$) and symmetric ($j = 2$) modes:

$$\Delta_j = \frac{12(1 - \cos p_j)^2 (k_3 + k'_3 R_j^4) + k_{40} + k'_{40} R_j^4}{\Omega_j k'_3 (1 + R_j^2)}$$

From this point one can obtain the value of the band-gap frequency and the threshold amplitude as shown above. In our case $\Omega_1(\pi) = 116.8 \text{ KHz}$ and $A_{th} = 0.4 \mu\text{m}$ with the driving frequency $\Omega = 117.8 \text{ KHz}$.